Surface Area

If a smooth parametric surface S is given by the equation $\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$, $(u,v) \in D$ and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is :

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA \text{ , where } \vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ , } \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Surface Area of the Graph of a function

For the special case of a surface S w/ equation $z = f(x,y)$, where x is in D and f has continuous partial derivatives, we take x and y as parameters.

The parametric equations are $x = x$, $y = y$, $z = f(x,y)$
or $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

So $\vec{r}_x = \hat{i} + \frac{\partial f}{\partial x}\hat{k}$, $\vec{r}_y = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$

and $\vec{r}_x \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$

Thus $\vec{r}_x \times \vec{r}_y = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$

and then the surface area formula becomes

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

and we derived this formula in Ch 15.6

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Ex Find the surface area of a sphere of radius a.

Soln $x = a\sin\phi\cos\theta$

$y = a\sin\phi\sin\theta$ and $D = \{(\phi,\theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$z = a\cos\phi$

First,

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\hat{i} + a^2\sin^2\phi\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k} = a\sin\phi \cdot \vec{r}(\theta,\phi).$$

Then, $|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$

$= \sqrt{a^4\sin^2\phi} = a^2\sin\phi$ (Since $\sin\phi \geq 0$)

Then $A = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi\,d\theta = a^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi = a^2(2\pi)(2) = 4\pi a^2.$

13.7 Surface Integrals

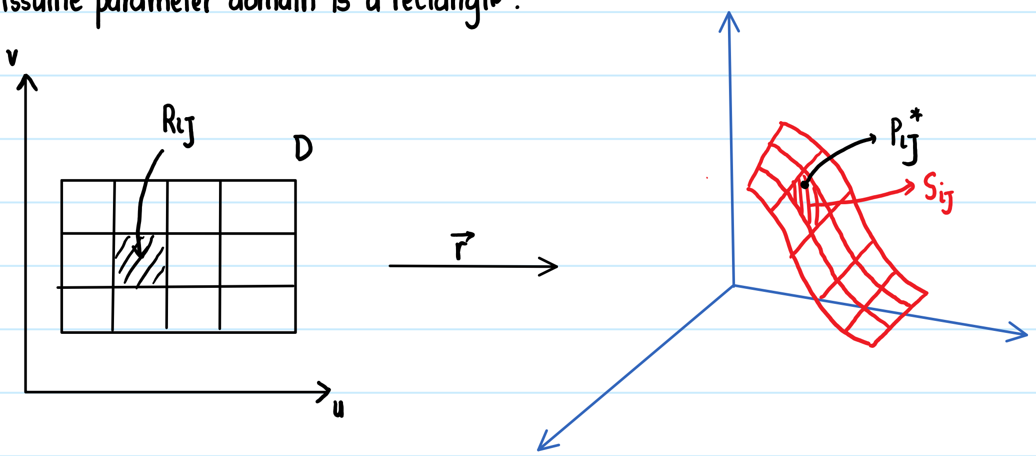
The relationship between surface integrals is the same as the relationship between line integrals and arc length.

We will define a surface integral of a function f of three variables defined over S in a such a way that, in the case when $f(x,y,z) = 1$, the value of the surface integral is equal to the surface area.

Surface Integrals

Suppose that a surface S has a vector equation
$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$, $(u,v) \in D$

Assume parameter domain is a rectangle.



Divide D into sub rectangles R_{ij} w/ dimensions Δu_i and Δv_j.
Then the surface S is divided into corresponding patches S_{ij}.
Evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

Then, $$\iint_S f(x,y,z)\,dS = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

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We can approximate $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u_i \Delta v_j$

where,

$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$, $\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are the tangent vectors at the corner of S_{ij}.

Then, $\displaystyle\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$

Ex Compute the surface integral $\displaystyle\iint_S x^2\, dS$, where S is the unit sphere $x^2+y^2+z^2=1$.

Soln We can parametrize sphere as $\vec{r}(\phi,\theta) = \sin\phi\cos\theta\, \hat{i} + \sin\phi\sin\theta\, \hat{j} + \cos\phi\, \hat{k}$ and we know from earlier example that $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$.

Then,
$$\iint_S x^2\, dS = \iint_D (\sin\phi\cos\theta)^2\, |\vec{r}_\phi \times \vec{r}_\theta|\, dA$$
$$= \int_0^{\pi}\int_0^{2\pi} \sin^2\phi\, \cos^2\theta\, \sin\phi\, d\theta\, d\phi = \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \cos^2\theta\, d\theta$$
$$= \int_0^{\pi} \sin\phi - \sin\phi\cos^2\phi\, d\phi \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$
$$= \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{4\pi}{3}$$

RmK If S is a piecewise-smooth surface i.e. a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ intersect only along their boundaries. Then,

$$\iint_S f(x,y,z)\, dS = \iint_{S_1} f(x,y,z)\, dS + \cdots + \iint_{S_n} f(x,y,z)\, dS$$

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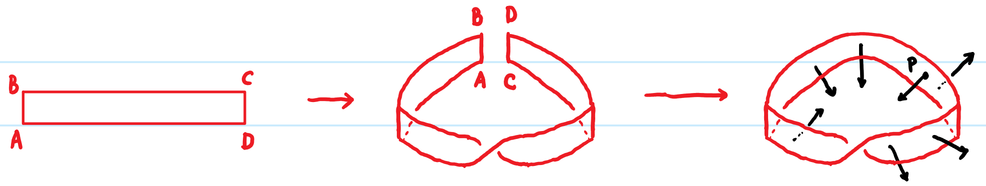
Oriented surfaces

To define surface integrals of vector fields, we need to rule out non-orientable surfaces.

Möbius Strip



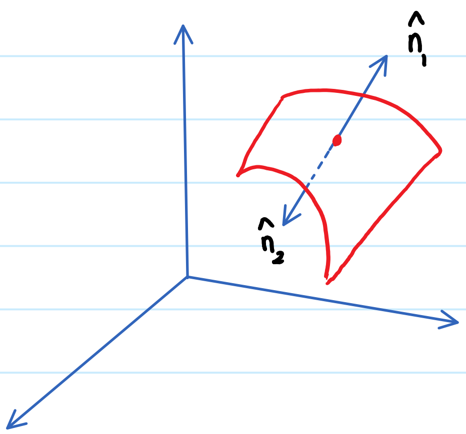
Take a rectangular piece of paper, give it a half twist, and tape the short edges together.

If an ant were to crawl on a Möbius strip, it would return to it's starting point having traversed the entire strip on both sides without ever having crossed an edge.
The Möbius strip really has one side. (Watch Video on Youtube)

http://mathinsight.org/moebius_strip_not_orientable

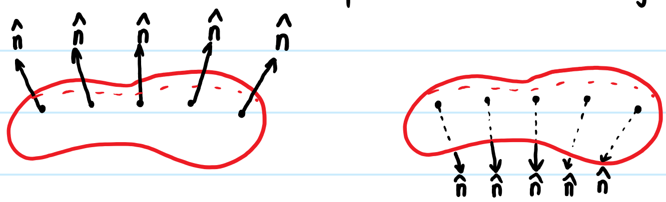
- We will consider only orientable (two-sided surfaces).

- Start w/ a surface S that has a tangent plane at every point (x,y,z) on S (except at boundary point). There are two unit normal vectors $\hat{n}_1$ and $\hat{n}_2 = -\hat{n}_1$.



If it is possible to choose a unit normal vector $\hat{n}$ at every such point so that $\hat{n}$ varies continuously over S, then S is called an oriented surface and given the choice of $\hat{n}$ provides S w/ an orientation.

Remark There are two possible orientations for any orientable surface.



- Let surface S be the graph of $z = g(x,y)$.
  We can associate to S a natural orientation given by the unit normal vector

$$\hat{n} = \frac{-\frac{\partial g}{\partial x}\hat{\imath} - \frac{\partial g}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Since $\hat{k}$-component is positive, this gives upward orientation of the surface.

- If S is a smooth surface given in parametric form by a vector function $\vec{r}(u,v)$, then it is automatically supplied w/ orientation of the unit normal vector

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ and the opposite orientation is given by $-\hat{n}$.

Ex For a sphere $x^2+y^2+z^2 = a^2$, we showed earlier that
$\vec{r}_\phi \times \vec{r}_\theta = a\sin\phi\ \vec{r}(\phi,\theta)$ and $|r_\phi \times r_\theta| = a^2 \sin\phi$

So the orientation is defined by $\hat{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times \vec{r}_\theta|} = \frac{1}{a}\vec{r}(\phi,\theta)$.

Note that $\hat{n}$ points in the same direction as the position vector, that is outward from the sphere.

The opposite orientation would have been obtained had we reversed the order of parameters since $\vec{r}_\theta \times \vec{r}_\phi = -\vec{r}_\phi \times \vec{r}_\theta$.

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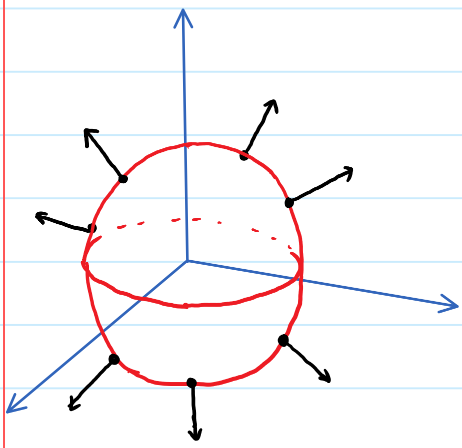
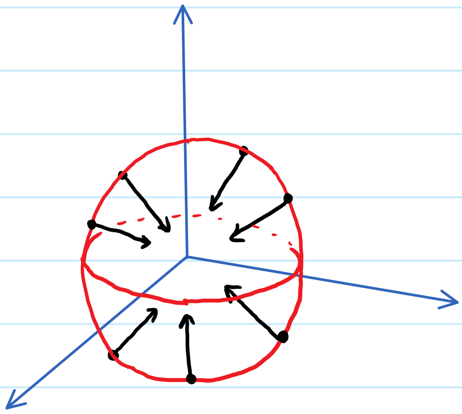


Positive orientation



Negative orientation.

http://mathinsight.org/parametrized_surface_orient

Convention For a closed surface, that is, a surface that is the boundary of a solid region E, the positive orientation is the one for which the normal vectors point outward from E, and inward pointing gives negative orientation.

Surface Integrals of vector fields

Defn If $\vec{F}$ is a continuous vector field defined on an oriented surface S w/ unit normal vector $\hat{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS$$

This integral is called the flux of $\vec{F}$ across S.

In other words, the surface integral of a vector field over S is equal to the surface integral of it's normal component over S.

If S is given by $\vec{r}(u,v)$, then

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$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS = \iint_D \left[\vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right] |\vec{r}_u \times \vec{r}_v|\, dA$$

where D is the parameter domain.

Thus, $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA$.

<u>Ex</u> Find the flux of the V.F. $\vec{F}(x) = z\hat{i} + y\hat{j} + x\hat{k}$ across the unit sphere $x^2+y^2+z^2=1$.

<u>Soln</u> $\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

Then, $\vec{F}(\vec{r}(\phi,\theta)) = \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta \rangle$ and

$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$

Therefore, $\vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi \times \vec{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$.

$= 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta$

Then the flux is,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta)\, dA = \int_0^{2\pi}\int_0^{\pi} 2\sin\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\, d\phi\, d\theta$$

$$= 2\int_0^{\pi} \sin^2\phi\cos\phi\, d\phi \int_0^{2\pi} \cos\theta\, d\theta + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{4\pi}{3}.$$

- If S is given by $z = g(x,y)$, we can think of x and y as parameters

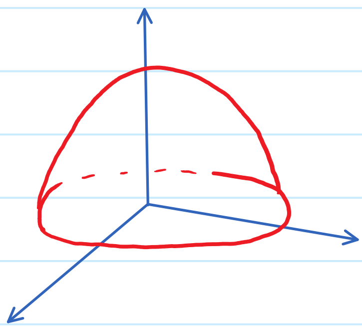
$$\vec{F}\cdot(\vec{r}_x \times \vec{r}_y) = \langle P,Q,R \rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle = -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R$$

Then, $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

The formula assumes upward orientation of S ; for downward orientation we multiply by -1.
Similar formula can be worked out for $x = h(y,z)$, $y = k(x,z)$.

<u>Ex</u> Evaluate the flux of $\vec{F}(x,y,z) = \langle y, x, z\rangle$ across S, where S is the boundary of the solid region E enclosed by the paraboloid $z = 1-x^2-y^2$ and the xy-plane.

<u>Soln</u>



The surface S consists of two pieces:
S_1 ≡ parabolic top.
S_2 ≡ circular bottom.

S is a closed surface, we use positive-outward orientation.

This means S_1 is oriented upward and projection of S onto the xy-plane is the disc $x^2+y^2 \leq 1$.

$P(x,y,z) = y$, $Q(x,y,z) = x$, $R(x,y,z) = z = 1-x^2-y^2$
and on S_1,

$$\frac{\partial g}{\partial x} = -2x \ , \ \frac{\partial g}{\partial y} = -2y$$

Then,

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA = \iint_D \left[-y(-2x) - x(-2y) + 1 - x^2 - y^2\right) dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\, r\,dr\,d\theta$$

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$$= \int_0^{2\pi}\int_0^1 r - r^3 + 4r^3\cos\theta\sin\theta \, dr\, d\theta = \int_0^{2\pi} \frac{1}{4} + \cos\theta\sin\theta \, d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}$$

The disc S_2 is oriented downward, so it's unit normal vector $\hat{n} = -\hat{k}$ and

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot(-\hat{k})\, dS = \iint_D (-z)\, dA = 0 \quad \text{as } z = 0 \text{ on } S_2.$$

Then, $$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

Applications

1) If $\vec{E}$ is an electric field, then the surface integral $\iint_S \vec{E}\cdot d\vec{S}$ is called the electric flux of $\vec{E}$ through S.

Important Law of Thermodynamics is Gauss's Law, which says that the net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \vec{F}\cdot d\vec{S}, \text{ where } \varepsilon_0 \text{ is a constant.}$$

2) Heat Flow : Suppose the temperature at a point (x,y,z) in a body is $u(x,y,z)$. Then the heat flow is defined as the vector field $\vec{F} = -K\nabla u$, where K is constant called the conductivity of the substance. The rate of heat flow across the surface S in the body is given by

$$\iint_S \vec{F}\cdot d\vec{S} = -K \iint_S \nabla u \cdot d\vec{S}$$